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Slowly oscillating sequences in locally normal Riesz spaces

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ABSTRACT

In the present paper, we are going to introduce and at the same time investigate the notion of slowly oscillating sequences, study on slowly oscillating compactness and slowly oscillating continuous functions in locally normal Riesz space. For this purpose, first of all, we are going to try to put forward some fundamental theorems about oscillating continuity, slowly oscillating compactness, sequential continuity and uniform continuity. Secondly, the newly obtained results in this paper can also be obtained with the definition of quasi-slowly oscillating and Δ-quasi-slowly oscillating sequences in terms of fuzzy points. Finally, most of the related theorems and lemmas are presented clearly.

1. Introduction

A Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by Riesz (1928). Riesz spaces have many applications in measure theory, operator theory and optimization. They have also some applications in economics (Aliprantis and Burkinshaw, 2003) and we refer to Albayrak and Pehlivan (2012), Alotaibi et al. (2014a, 2014b), Luxemburg and Zaanen (1971), Mohiuddine et al. (2012, 2013), Roberts (1952), Zannen (1997) for more details.

A real valued function is continuous on the set of real numbers if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function and the idea of compactness in terms of sequences, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: forward continuity (Cakalli, 2011a), slowly oscillating continuity (Cakalli, 2008; Dik and Canak, 2010; Hazarika, 2016; Tripathy and Baruah, 2010; Vallin, 2011), ideal ward continuity (Hazarika, 2014b; Hazarika and Esi, 2016a), ϕ-statistical ward continuity (Hazarika and Esi, 2016a), ϕ-ideal ward continuity (Hazarika and Esi, 2016b). The concept of a Cauchy sequence involves far more than that the distance between successive terms is tending to zero. Nevertheless, sequences which satisfy this weaker property are interesting in their own right. A sequence (x_n) of points in $\mathbb{R}$ is called quasi-Cauchy if (Δx_n) is a null sequence where $\Delta x_n = x_{n+1} - x_n$. Burton and Coleman (2010) named these sequences as "quasi-Cauchy" and Cakalli (2011b) used the term "ward convergent to 0" sequences. In terms of quasi-Cauchy we restate the definitions of ward compactness and ward continuity as follows: a function f is ward continuous if it preserves quasi-Cauchy sequences, i.e. $(f(x_n))$ is quasi-Cauchy whenever (x_n) is, and a subset E of $\mathbb{R}$ is ward compact if any sequence $\mathrm{x} = (x_n)$ of points in E has a quasi-Cauchy subsequence $\mathrm{z} = (z_k) = (x_{n_k})$ of the sequence x.

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2. Preliminaries and notations

It is known that a sequence (x_n) of points in $\mathbb{R}$, the set of real numbers, is slowly oscillating if

$$\lim_{\lambda \to 1^+} \overline{\lim}_n \max_{n+1 \le k \le [\lambda n]} |x_k - x_n| = 0$$

where, $[\lambda n]$ denotes the integer part of λn. This is equivalent to the following if $(x_m - x_n) \to 0$ whenever $1 \le \frac{m}{n} \to 1$ as $m, n \to \infty$. Using $\varepsilon > 0$ and δ this is also equivalent to the case when for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that $|x_m - x_n| < \varepsilon$ if $n \ge N(\varepsilon)$ and $n \le m \le (1+\delta)n$ (Cakalli, 2008).

A function defined on a subset E of $\mathbb{R}$ is called slowly oscillating continuous if it preserves slowly oscillating sequences, i.e. $(f(x_n))$ is slowly oscillating whenever (x_n) is.

Connor and Grosse-Erdmann (2003) gave sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, space of all sequences, denoted by c_G, into $\mathbb{R}$. A sequence $\mathrm{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathrm{x} \in c_G$ and $G(\mathrm{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathrm{x} = \lim_n x_n$ on the linear space c.

A method G is called regular if every convergent sequence $\mathrm{x} = (x_n)$ is G-convergent with $G(\mathrm{x}) = \lim \mathrm{x}$. A method is called subsequential if whenever x is G-convergent with $G(\mathrm{x}) = \ell$, then there is a subsequence (x_{n_k}) of x with $\lim_k x_{n_k} = \ell$ (Cakalli, 2011c).

Let X be a real vector space and $\leq$ be a partial order on this space. Then X is said to be an *ordered vector space* if it satisfies the following properties:

(i) if $x, y \in X$ and $y \leq x$, then $y + z \leq x + z$ for each $z \in X$.
(ii) if $x, y \in X$ and $y \leq x$, then $ay \leq ax$ for each $a \geq 0$.

If, in addition, X is a lattice with respect to the partial ordered, then X is said to be a *Riesz space* (or a *vector lattice*) (Zannen, 1997).

For an element x of a Riesz space X, the *positive part* of x is defined by $x^+ = x \vee \overline{0} = \sup\{x, \overline{0}\}$, the *negative part* of x by $x^- = -x \vee \overline{0}$ and the *absolute value* of x by $|x| = x \vee (-x)$, where $\overline{0}$ is the zero element of X.

A subset S of a Riesz space X is said to be *normal* if $y \in S$ and $|x| \leq |y|$ implies $x \in S$.

A *topological vector space* (X, τ) is a vector space X which has a topology (linear) τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f: X \times X \to X$ defined by $f(x, y) = x + y$ is continuous on $X \times X$, and continuity of scalar multiplication means that the function $f: \mathbb{R} \times X \to X$ defined by $f(a, x) = ax$ is continuous on $\mathbb{R} \times X$.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties:

(1) Each $Y \in N$ is a *balanced set,* that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.
(2) Each $Y \in N$ is an *absorbing set*, that is, for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.
(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally normal* if τ has a base at zero consisting of solid sets. A *locally normal Riesz space* (X, τ) is a Riesz space equipped with a locally normal topology τ.

Recall that a first countable space is a topological space satisfying the "first axiom of countability". Specifically, a space X is said to be first countable if each point has a countable neighborhood basis (local base). That is, for each point x in X there exists a sequence $V_1, V_2, \cdots$ of open neighborhoods of x such that for any open neighborhood V of x there exists an integer j with V_j contained in V.

Throughout the article, the symbol N_{nor} we will denote any base at zero consisting of normal sets and satisfying the conditions (1), (2) and (3) in a locally normal topology. Also (X, τ) a locally normal Riesz space (in short LNRS) and $\mathbb{N}$ and $\mathbb{R}$ will denote the set of all positive integers, and the set of all real numbers, respectively. We will use boldface letters x, y, z, ... for sequences $\mathrm{x} = (x_n)$, $\mathrm{y} = (y_n)$, $\mathrm{z} = (z_n)$, ... of points in X.

3. Slowly oscillating sequences in LNRS

In this section we introduce the concepts of slowly oscillating continuity and slowly oscillating compactness in LNRS and establish some interesting results related to these notions.

A sequence $\mathrm{x} = (x_n)$ of points in X is called quasi-Cauchy if for each τ-neighborhood V of zero, there exists an $m_0 \in \mathbb{N}$ such that $x_{n+1} - x_n \in V$ for $\geq m_0$. It is clear that Cauchy sequences are slowly oscillating not only the real case but also in the LNRS setting. It is easy to see that any slowly oscillating sequence of points in X is quasi-Cauchy and therefore Cauchy sequence is quasi-Cauchy. The converses are not always true. There are quasi-Cauchy sequences which are not Cauchy. There are quasi-Cauchy sequences which are not slowly oscillating. Any subsequence of Cauchy sequence is Cauchy. The analogous property fails for quasi-Cauchy sequences and slowly oscillating sequences as well.

Now we introduce the notion of slowly oscillating sequences and slowly oscillating continuity in LNRS.

Definition 3.1: *A sequence $x = (x_n)$ of points in X is said to be slowly oscillating if for each τ-neighborhood V of zero, there exist $\delta = \delta(V) > 0$ and $m = m(V)$ such that*

$$x_k - x_n \in V \text{ for } n \geq m(V) \text{ and } n \leq k \leq (1 + \delta)n.$$

It is clear that a convergent sequence is slowly oscillating, since every convergent sequence is a Cauchy sequence, and any slowly oscillating sequence is quasi-Cauchy, but the converse need not to be true in general. For examples, $(\sum_{n=1}^{\infty} \frac{1}{n})$, $(\ln n)$, $(\ln \ln n)$ are slowly oscillating, but not Cauchy. The sequence $(\sum_{k=1}^{n} \frac{1}{k})$ is quasi-Cauchy, but not slowly oscillating.

Definition 3.2: *A function f defined on a subset E of X is called slowly oscillating continuous if it transforms slowly oscillating sequences to slowly oscillating sequences of points in E, that is, $(f(x_n))$ is slowly oscillating whenever (x_n) is slowly oscillating sequences of points in E.*

Theorem 3.3: *If f is slowly oscillating continuous on a subset E of X then it is continuous on E in the ordinary sense.*

Proof. Suppose that f is slowly oscillating continuous on E and let (x_n) be any convergent sequence of points in E with $\lim_{n\to\infty} x_n = x_0$. Then the sequence

$$(y_n) = (x_1, x_0, x_2, x_0, \ldots, x_{n-1}, x_0, x_n, x_0, \ldots)$$

is also convergent to x_0 and hence (y_n) is slowly oscillating. Since f is slowly oscillating continuous, the sequence

$$(f(y_n)) = (f(x_1), f(x_0), f(x_2), f(x_0), \ldots, f(x_{n-1}), f(x_0), f(x_n), f(x_0), \ldots)$$

is also slowly oscillating. Hence $(f(y_n))$ is a quasi-Cauchy sequence. Now it follows that if for each τ-neighborhood V of zero, there exists $m = m(V)$ such that

$$f(x_n) - f(x_0) \in V \text{ for } n \geq m$$

this completes the proof of theorem.

Theorem 3.4: *If f and g are slowly oscillating continuous functions on a subset E of X. Then $f + g$ is slowly oscillating continuous in E.*

Proof. Let f and g be slowly oscillating continuous functions on a subset E of X. To prove that $f + g$ is slowly oscillating continuous on E. Let $\mathbf{x} = (x_n)$ is any slowly oscillating sequence in E. Then $(f(x_n))$ and $(g(x_n))$ are slowly oscillating sequences. Since $(f(x_n))$ and $(g(x_n))$ are slowly oscillating sequences, if for each τ-neighborhood V of zero, there exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W + W \subseteq Y$, there exists $\delta > 0$ and a positive integer n_1 such that

$$f(x_k) - f(x_n) \in W \text{ and } g(x_k) - g(x_n) \in W$$

for all $n \geq n_1$ and $n \leq k \leq (1+\delta)n$. Therefore, if $n \geq n_1$ and $n \leq k \leq (1+\delta)n$, we have

$$\begin{aligned}(f+g)(x_k) - (f+g)(x_n) &= f(x_k) - f(x_n) + g(x_k) - g(x_n) \\ &\in W + W \subseteq Y \subseteq V.\end{aligned}$$

This implies that $(f+g)(x_k) - (f+g)(x_n) \in V$ for all $n \geq n_1$ and $n \leq k \leq (1+\delta)n$. This complets the proof of the theorem.

In Vallin (2011), it was proved that a slowly oscillating continuous function is uniformly continuous on $\mathbb{R}$. We see that is also the case that any slowly oscillating continuous function on a connected subset of X is uniformly continuous.

Theorem 3.5: *If f is a uniformly continuous function defined on a subset E of X, then it is slowly oscillating continuous on E.*

Proof. Let f be uniformly continuous function and $\mathbf{x} = (x_n)$ be any slowly oscillating sequence in E. Let W be a τ-neighborhood of zero. Since f is uniformly continuous on E, then there exists a τ-neighborhood V of zero such that $f(x) - f(y) \in W$ whenever $x - y \in V$. Since (x_n) is slowly oscillating, for the same τ-neighborhood W of zero, there exist $m = m(V)$ and $\delta = \delta(V) > 0$ such that $x_k - x_n \in V$ for $n \geq m(V)$ and $n \leq k \leq (1+\delta)n$. Hence we have $f(x_k) - f(x_n) \in W$ whenever $n \geq m(V)$ and $n \leq k \leq (1+\delta)n$. It follows that $(f(x_n))$ is slowly oscillatting. This completes the proof of theorem.

Definition 3.6: *A sequence (x_n) of points in X is called Cesáro slowly oscillating if (t_n) is slowly oscillating, where $t_n = \frac{1}{n}\sum_{k=1}^{n} x_k$, is the Cesáro means of the sequence (x_n). Also a function f defined on a subset E of X is called Cesáro slowly oscillating continuous if it preserves Cesáro slowly oscillating sequences of points in E.*

By using the similar argument used in proof of Theorem 3.5, we immediately have the following result.

Theorem 3.7: *If f is a uniformly continuous on a subset E of X and (x_n) is a slowly oscillating sequence in E, then $(f(x_n))$ is Cesáro slowly oscillating.*

Definition 3.8: *A sequence of functions (f_n) defined on a subset E of X is said to be uniformly convergent to a function f if for each τ-neighborhood V of zero, there exists an integer $n_0 = n_0(V)$ such that $f_n(x) - f(x) \in V$ for all $n \geq n_0$ and $x \in E$.*

Theorem 3.9: *If (f_n) is a sequence of slowly oscillating continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f on E, then f is slowly oscillating continuous on E.*

Proof. Let (x_n) be any slowly oscillating sequence of points in E. By uniform convergence of (f_n), if for each τ-neighborhood V of zero, there exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W + W + W \subseteq Y$. Then there exists $n_1 = n_1(V)$ such that

$$f_n(x) - f(x) \in W$$

for each $x \in E$ and for all $n \geq n_1$. Also since f_{n_1} is slowly oscillating continuous, there exist $n_2 = n_2(V)$ and $\delta = \delta(V) > 0$ such that

$$f_{n_1}(x_k) - f_{n_1}(x_n) \in W$$

whenever $n \geq n_2$ and $n \leq k \leq (1+\delta)n$. Let $m = m(V) = \max\{n_1(V), n_2(V)\}$. Therefore if $n \geq m$ and $n \leq k \leq (1+\delta)n$ we have

$$f(x_k) - f(x_n) = f(x_k) - f_{n_1}(x_k) + f_{n_1}(x_k) - f_{n_1}(x_n) + f_{n_1}(x_n) - f(x_n) \in W + W + W \subseteq Y \subseteq V.$$

Thus it implies that $f(x_k) - f(x_n) \in V$ if $n \geq m$ and $n \leq k \leq (1+\delta)n$. It follows that $(f(x_n))$ is a slowly oscillating sequence of points in E which completes the proof of theorem.

Using the same techniques as in the Theorem 3.9, the following result can be obtained easily.

Theorem 3.10: *If (f_n) is a sequence of Cesáro slowly oscillating continuous functions defined on a subset E of X and (f_n) is uniformly convergent to a function f on E, then f is Cesáro slowly oscillating continuous on E.*

Theorem 3.11: *The set of all slowly oscillating continuous functions defined on a subset E of X is a closed subset of all continuous functions on E, that is $\overline{SOC(E)} = SOC(E)$, where $SOC(E)$ is the set of all slowly oscillating continuous functions defined on E and $\overline{SOC(E)}$ denotes the set of all cluster points of $SOC(E)$.*

Proof. Let f be any element of $\overline{SOC(E)}$. Then there exists a sequence of points in $SOC(E)$ such that $\lim_{k\to\infty} f_k = f$. To show that f is slowly oscillating sequence on E. Now let (x_n) be any slowly oscillating sequence in E. Let V be an arbitrary τ-neighborhood of zero. There exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W + W + W \subseteq Y$. Since (f_k) converges to f, there exists a positive integer n_1 such that for all $x \in E$ and for all $n \geq n_1$, $f(x) - f_n(x) \in W$. Also since f_{n_1} is slowly oscillating continuous, there exists an integer $n_2 = n_2 > n_1$ and $\delta = \delta(V) > 0$ such that

$$f_{n_1}(x_k) - f_{n_1}(x_n) \in W$$

whenever $n \geq n_2$ and $n \leq k \leq (1+\delta)n$. Hence, for all $n \geq n_1$ and $n \leq k \leq (1+\delta)n$ we have

$$f(x_k) - f(x_n) = f(x_k) - f_{n_1}(x_k) + f_{n_1}(x_k) - f_{n_1}(x_n) + f_{n_1}(x_n) - f(x_n) \in W + W + W \subseteq Y \subseteq V.$$

Thus it implies that $f(x_k) - f(x_n) \in V$ for all $n \geq n_1$ and $n \leq k \leq (1+\delta)n$. Thus f is slowly oscillating continuous function on E and this completes the proof of theorem.

Corollary 3.12: *The set of all slowly oscillating continuous functions defined on a subset E of X is a complete subspace of the space of all continuous functions on E.*

Next we define the concept of slowly oscillating compactness in LNRS.

Definition 3.13: *A subset E of X is called slowly oscillating compact if any sequence of points in E has a slowly oscillating subsequence.*

We see that any compact subset of X is slowly oscillatting compact, union of two slowly oscillating compact subsets of X is slowly oscillating compact. Any subset of slowly oscillating compact set is also slowly oscillating compact and so intersection of any slowly oscillating compact subsets of X is slowly oscillating compact.

Theorem 3.14: *A slowly oscillating continuous image of a slowly oscillating compact subset of X is slowly oscillating compact.*

Proof. Let f be a slowly oscillating continuous function on X and E be a slowly oscillating compact subset of X. Let $\mathbf{y} = (y_n)$ be a sequence of points in $f(E)$. Then we can write $y_n = f(x_n)$ where (x_n) is sequence of points in E for each $n \in \mathbb{N}$. Since E is slowly oscillating compact, there is a slowly oscillating subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of (x_n). Then, slowly oscillating continuity of f implies that $f(z_k)$ is a slowly oscillating subsequence of $f(x_n)$. Hence $f(E)$ is slowly oscillating compact.

We say that a subset E of X is called Cauchy compact if any sequence of points of E has a Cauchy subsequence. We see that any Cauchy compact subset of X is also slowly oscillating compact and slowly oscillating continuous image of any Cauchy compact subset of X is Cauchy compact.

Corollary 3.15: *For any regular subsequential method G, if E is G-sequentially compact subset of X, then it is slowly oscillating compact.*

Proof. The proof of the result follows from the regularity and subsequence property of G.

Theorem 3.16: *Let E be a slowly oscillating compact subset of X and let $f: E \to X$ be a slowly oscillating continuous function. Then f is uniformly continuous on E.*

Proof. Suppose that f is not uniformly continuous on E. Let V be an arbitrary τ-neighborhood of zero. There exists a $Y \in N_{nor}$ such that $Y \subseteq V$. Choose $W \in N_{nor}$ such that $W + W + W \subseteq Y$. Let (x_n) and (y_n) be sequences of points in E. Let m be a positive integer such that $x_n - y_n \in W$ for all $n \geq m$, but:

$$f(x_n) - f(y_n) \notin V \text{ for all } n \geq m. \tag{3.1}$$

Since E is slowly oscillating compact, there is a slowly oscillating subsequence (x_{n_k}) of (x_n). It is clear from the result

$y_{n_k} - y_{n_m} = y_{n_k} - x_{n_k} + x_{n_k} - x_{n_m} + x_{n_m} - y_{n_m} \in W + W + W \subseteq Y$

that the corresponding subsequence (y_{n_k}) of (y_n) is also slowly oscillating. Then from the result (3.1) we observe that the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ are not slowly oscillating. This contradiction completes the proof of the theorem.

Corollary 3.17: *Let E be a slowly oscillating compact subset of X and let $f: E \to X$ be a slowly oscillating continuous function. Then f is uniformly continuous if and only if it is slowly oscillating continuous.*

Corollary 3.18: *A real valued function defined on a bounded subset of $\mathbb{R}$ is uniformly continuous if and only if it is slowly oscillating continuous.*

Proof. The proof of the result follows from the fact that totally boundedness coincides with slowly oscillating compactness and boundedness coincides with totally boundedness in $\mathbb{R}$.

Kostyrko et al. (2000) introduced the notion of ideal convergence which is a generalization of statistical convergence (Fast, 1951; Fridy, 1985) based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$.

A family of sets $I \subset P(\mathbb{N})$ (the power sets of $\mathbb{N}$) is said to be an *ideal* on $\mathbb{N}$ if and only if $\phi \in I$ for each $A, B \in I$, we have $A \cup B \in I$ for each $A \in I$ and each $B \subset A$, we have $B \in I$. A non-empty family of sets $F \subset P(\mathbb{N})$ is said to be a *filter* on $\mathbb{N}$ if and only if $\phi \notin F$ for each $A, B \in F$, we have $A \cap B \in F$ each $A \in F$ and each $B \supset A$, we have $B \in F$. An ideal I is called *non-trivial* ideal if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly $I \subset P(\mathbb{N})$ is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A: A \in I\}$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subset P(\mathbb{N})$ is called *admissible* if and only if $\{\{n\}: n \in \mathbb{N}\} \subset I$. Throughout we assume I is a non-trivial admissible ideal in $\mathbb{N}$.

A sequence $\mathrm{x} = (x_n)$ of points in a locally normal Riesz space X is said to be ideally convergent to $x_0 \in X$ if for every τ-neighborhood V of zero, the set $\{n \in \mathbb{N}: x_n - x_0 \notin V\} \in I$. In this case we write $x_n \xrightarrow{I_\tau} \ell$ i.e. $I_\tau\text{-lim}x_n = \ell$ (for details see Hazarika (2014a)).

Definition 3.19: *Let (X, τ_1) and (Y, τ_2) be LNR spaces and $E \subset X$. A function $f: E \to Y$ is called ideally continuous at a point $x_0 \in E$ if $x_n \xrightarrow{I_{\tau_1}} x_0$ in E implies*

$f(x_n) \xrightarrow{I_{\tau_2}} f(x_0)$ in Y.

An element x_0 in X is called an ideal limit point of a subset E of X if there is an E-valued sequence of points with ideal limit x_0. It follows that the set of all ideal limit points of E is equal to the set of all limit points of E in the ordinary sense. An element x_0 in X is called an ideal accumulation point of a subset E if it is an ideal limit point of the set $E - \{x_0\}$. The set of all ideal accumulation points of E is equal to the set of all accumulation points of E in the ordinary sense.

A function f on X is said to have an ideally sequential limit at a point x_0 of X if the image sequence $(f(x_n))$ is ideally convergent to x_0 for any ideally convergent sequence $\mathrm{x} = (\mathrm{x}_n)$ with ideal limit x_0 and a function f is to be ideally sequentially continuous at a point x_0 of X if the sequence $(\mathrm{f}(\mathrm{x}_n))$ is ideally convergent to $\mathrm{f}(\mathrm{x}_0)$ for any ideally convergent sequence $\mathrm{x} = (\mathrm{x}_n)$ with ideal limit x_0 (for details see Cakalli and Hazarika (2012)).

Lemma 3.20: *A function f on X has an ideally sequential limit at a point x_0 of X if and only if it has an ideal limit at a point x_0 of X in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence (also see Cakalli and Hazarika (2012)).

Theorem 3.21: *A function f on X is ideally sequentially continuous at a point x_0 of X if and only if it is continuous at a point x_0 in ordinary sense.*

Proof. The proof follows from the fact that any ideally convergent sequence has a convergent subsequence and from the above lemma.

Theorem 3.22: *If a function is slowly oscillating continuous on a subset E of X, then it is ideally sequentially continuous on E.*

Proof. Let f be any slowly oscillating continuous on E. By Theorem 3.3, we have f is continuous on E. Also from Theorem 3.21, we see that f is ideally sequentially continuous on E. This completes the proof.

Theorem 3.23: *If a function is δ-ward continuous on a subset E of X, then it is ideally sequentially continuous on E.*

Proof. Let f be any δ-ward continuous function on E. It follows from Corollary 2 in Cakalli (2011d) that f is continuous. By Theorem 3.21 we obtain that f is ideally sequentially continuous on E. This completes the proof of the theorem.

4. Conclusions

In this paper, the concept of slowly oscillating continuity and slowly oscillating compactness in locally normal Riesz spaces are introduced and investigated. In this investigation we have obtained theorems related to slowly oscillating continuity, slowly oscillating compactness, sequential continuity and uniform continuity. Finally, we note that the results of this paper can be obtained by defining the ideas of quasi-slowly oscillating and Δ-quasi-slowly oscillating sequences of fuzzy points (for fuzzy setting, we refer to Cakalli and Das (2009), Hazarika (2014c, 2013a, 2013b), and Hazarika and Savas, 2011)).

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